Indian Statistical Institute, Bangalore

M.Math II Year, Second Semester

Semestral Examination

Advanced Functional Analysis

Time: 3 hours May 02, 2011 Instructor: T.S.S.R.K.Rao

Total Score : $10 \times 6 = 60$

Answer all questions. Show all work.

1. Consider $\mathbb{R}^n$, equipped with the norms, $\| (x_1 \cdots x_n) \|_1 = \sum_{i=1}^{n} | x_i |$ and $\| (x_1, \cdots, x_n) \|_\infty = \max_{1 \leq i \leq n} | x_i |$. Let $T : (\mathbb{R}^n, \|\|_1) \longrightarrow (\mathbb{R}^n, \|\|_\infty)$ be a linear map that in also one - one. Show that T is an isomorphism.

2. Let X, Y, Z be completely metrizable TVS. Suppose $B : X \times Y \to Z$ is bilinear and separately continuous. Show that B is continuous.

3. Let X be a locally convex topological vector space. Let V be a closed, convex, balanced nbhd of zero. Show that: ${}^0(V^0) = V$.

4. Let $l^1 = \{\{\alpha_n\}_{n \geq 1} : \sum | \alpha_n | \quad < \infty\}$, equipped with the usual norm. Let $A \subset l^1$ be bounded in the weak topology. Show that A is bounded in the norm topology.

5. Let K be a compact convex set in a $LCTVS$, X. Suppose K has only finitely many extreme points. Show that K is the convex hull of the set of extreme points.

6. Let $f : [0,1] \to \mathbb{R}^+$ be a continuous, affine function (i.e., $f(\lambda t + (1 - \lambda)s) = \lambda f(t) + (1 - \lambda) f(s)$ for $\lambda, s, t \in [0,1]$). Show that $\sup_{[0,1]} f = f(0)$ or $f(1)$.

7. Let K be compact convex set in a $LCTVS$ X. Let $a : K \to \mathbb{R}$ be an affine continuous function. Suppose $a = 0$ at all extreme points of K. Show that $a \equiv 0$.

8. Let (l, σ, m) be a finite measure space. Let X be a Banach space. Let $f, g : l \to X$ be strongly $m-$ measurable. Suppose $\forall x^* \varepsilon X^*, x^* \circ f = x^* \circ g \ a \cdot e$. Show that $f = g \ a \cdot e$.

9. Let (l, σ, m) be a probability space. Let X be a Banach space. Let $f : l \to X$ be a Bochner integrable function. Let $\{E_n\}_{n \geq 1} \subset \sigma$ be a pair-wise disjoint sequence. Show that $\int\limits_{UE_n} f dm = \sum \int\limits_{E_n} f dm$.

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10. Let Σ be the Borel $\sigma-$ field on $[0,1]$ and λ, be the Lebesgue measure. Consider $\mathbb{R}^n$ equipped with the maximum norm. Let $f : [0,1] \to \mathbb{R}^n$ be such that for each coordinate functional e_i, $e_i \circ f \in L^1(\lambda)$. Show that f is Bochner integrable.